Analysis of the Sallen-Key Architecture

Application Report

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Analysis of the Sallen-Key Architecture

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ABSTRACT

This application report discusses the Sallen-Key architecture. The report gives a general overview and derivation of the transfer function, followed by detailed discussions of low-pass and high-pass filters, including design information, and ideal and non-ideal operation. To illustrate the limitations of real circuits, data on low-pass and high-pass filters using the Texas Instruments THS3001 is included. Finally, component selection is discussed.

1 Introduction

Figure 1 shows a two-stage RC network that forms a second order low-pass filter. This filter is limited because its Q is always less than 1/2. With R1=R2 and C1=C2, Q=1/3. Q approaches the maximum value of 1/2 when the impedance of the second RC stage is much larger than the first. Most filters require Qs larger than 1/2.
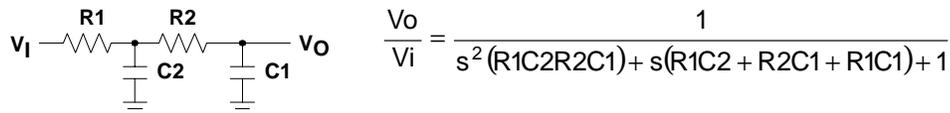

$$\frac{Vo}{Vi} = \frac{1}{s^2(R1C2R2C1) + s(R1C2 + R2C1 + R1C1) + 1}$$

Figure 1. Basic Second Order Low-Pass Filter

Larger Qs are attainable by using a positive feedback amplifier. If the positive feedback is controlled—localized to the cut-off frequency of the filter—almost any Q can be realized, limited mainly by the physical constraints of the power supply and component tolerances. Figure 2 shows a unity gain amplifier used in this manner. Capacitor C2, no longer connected to ground, provides a positive feedback path. In 1955, R. P. Sallen and E. L. Key described these filter circuits, and hence they are generally known as Sallen-Key filters.
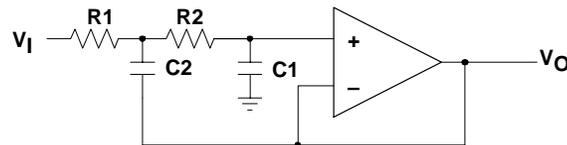
Figure 2. Unity Gain Sallen-Key Low-Pass Filter

The operation can be described qualitatively:

- At low frequencies, where C1 and C2 appear as open circuits, the signal is simply buffered to the output.
- At high frequencies, where C1 and C2 appear as short circuits, the signal is shunted to ground at the amplifier's input, the amplifier amplifies this input to its output, and the signal does not appear at Vo.
- Near the cut-off frequency, where the impedance of C1 and C2 is on the same order as R1 and R2, positive feedback through C2 provides Q enhancement of the signal.

2 Generalized Circuit Analysis

The circuit shown in Figure 3 is a generalized form of the Sallen-Key circuit, where generalized impedance terms, Z, are used for the passive filter components, and R3 and R4 set the pass–band gain.
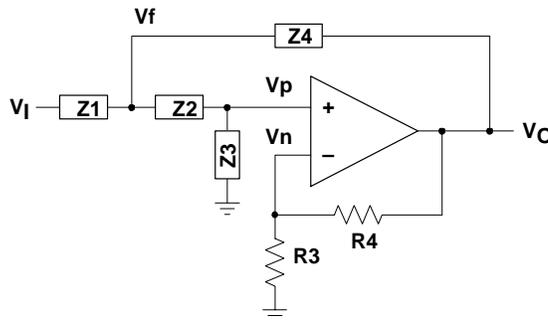

Figure 3. Generalized Sallen-Key Circuit

To find the circuit solution for this generalized circuit, find the mathematical relationships between Vi, Vo, Vp, and Vn, and construct a block diagram.

KCL at Vf:

$$Vf\left(\frac{1}{Z1} + \frac{1}{Z2} + \frac{1}{Z4}\right) = Vi\left(\frac{1}{Z1}\right) + Vp\left(\frac{1}{Z2}\right) + Vo\left(\frac{1}{Z4}\right) \tag{1}$$

KCL at Vp:

$$Vp\left(\frac{1}{Z2} + \frac{1}{Z3}\right) = Vf\left(\frac{1}{Z2}\right) \Rightarrow Vf = Vp\left(1 + \frac{Z2}{Z3}\right) \tag{2}$$

Substitute Equation (2) into Equation (1) and solve for Vp:

$$Vp = Vi\left(\frac{Z2Z3Z4}{Z2Z3Z4 + Z1Z2Z4 + Z1Z2Z3 + Z2Z2Z4 + Z2Z2Z1}\right) +$$

$$Vo\left(\frac{Z1Z2Z3}{Z2Z3Z4 + Z1Z2Z4 + Z1Z2Z3 + Z2Z2Z4 + Z2Z2Z1}\right) \tag{3}$$

KCL at Vn:

$$Vn\left(\frac{1}{R3} + \frac{1}{R4}\right) = Vo\left(\frac{1}{R4}\right) \Rightarrow Vn = Vo\left(\frac{R3}{R3 + R4}\right) \tag{4}$$

2.1 Gain Block Diagram

By letting: a(f) = the open-loop gain of the amplifier, $b = \left(\dfrac{R3}{R3 + R4}\right)$,

$$c = \frac{Z2Z3Z4}{Z2Z3Z4 + Z1Z2Z4 + Z1Z2Z3 + Z2Z2Z4 + Z2Z2Z1},$$

$$d = \frac{Z1Z2Z3}{Z2Z3Z4 + Z1Z2Z4 + Z1Z2Z3 + Z2Z2Z4 + Z2Z2Z1},$$

and Ve = Vp − Vn, the generalized Sallen-Key filter circuit is represented in gain-block form as shown in Figure 4.
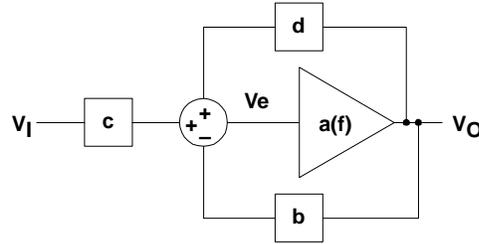
Figure 4. Gain-Block Diagram of the Generalized Sallen-Key Filter

From the gain-block diagram the transfer function can be solved easily by observing, Vo = a(f)Ve and Ve = cVi + dVo − bVo. Solving for the generalized transfer function from gain block analysis gives:

$$\frac{Vo}{Vi} = \left(\frac{c}{b}\right)\left[\frac{1}{1 + \dfrac{1}{a(f)b} - \dfrac{d}{b}}\right] \tag{5}$$

2.2 Ideal Transfer Function

Assuming a(f)b is very large over the frequency of operation, $\dfrac{1}{a(f)b} \approx 0$, the ideal transfer function from gain block analysis becomes:

$$\frac{Vo}{Vi} = \left(\frac{c}{b}\right)\left[\frac{1}{1 - \dfrac{d}{b}}\right] \tag{6}$$

By letting $\dfrac{1}{b} = K$, $c = \dfrac{N1}{D}$, and $d = \dfrac{N2}{D}$, where N1, N2, and D are the numerators and denominators shown above, the ideal equation can be rewritten as:

$\dfrac{Vo}{Vi} = \left[\dfrac{K}{\dfrac{D}{N1} - \dfrac{K \cdot N2}{N1}}\right].$ Plugging in the generalized impedance terms gives the ideal transfer function with impedance terms:

$$\frac{Vo}{Vi} = \frac{K}{\dfrac{Z1Z2}{Z3Z4} + \dfrac{Z1}{Z3} + \dfrac{Z2}{Z3} + \dfrac{Z1(1-K)}{Z4} + 1} \tag{7}$$

3   Low-Pass Circuit

The standard frequency domain equation for a second order low-pass filter is:

$$H_{LP} = \frac{K}{-\left(\frac{f}{fc}\right)^2 + \frac{jf}{Qfc} + 1} \tag{8}$$

Where fc is the corner frequency (note that fc is the breakpoint between the pass band and stop band, and is not necessarily the −3 dB point) and Q is the quality factor. When f<<fc Equation (8) reduces to K, and the circuit passes signals multiplied by a gain factor K. When f=fc, Equation (8) reduces to −jKQ, and signals are enhanced by the factor Q. When f>>fc, Equation (8) reduces to $-K\left(\frac{fc}{f}\right)^2$, and signals are attenuated by the square of the frequency ratio. With attenuation at higher frequencies increasing by a power of 2, the formula describes a second order low-pass filter.

Figure 5 shows the Sallen-Key circuit configured for low-pass:

$$Z1 = R1, \;\; Z2 = R2, \;\; Z3 = \frac{1}{sC1},$$

$$Z4 = \frac{1}{sC2}, \;\; \text{and} \;\; K = 1 + \frac{R4}{R3}.$$
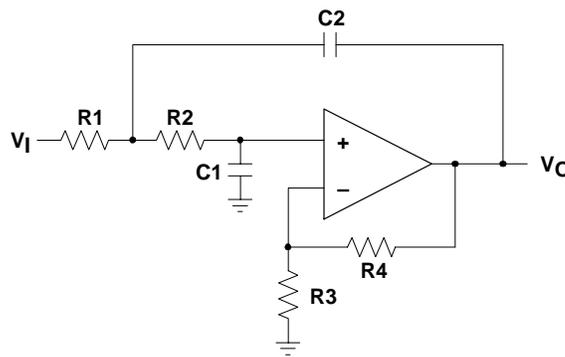
Figure 5. Low-Pass Sallen-Key Circuit

From Equation (7), the ideal low-pass Sallen-Key transfer function is:

$$\frac{Vo}{Vi}(lp) = \frac{K}{s^2(R1R2C1C2) + s(R1C1 + R2C1 + R1C2(1-K)) + 1} \tag{9}$$

By letting

$$s = j2\pi f, \;\; fc = \frac{1}{2\pi\sqrt{R1R2C1C2}}, \;\; \text{and} \;\; Q = \frac{\sqrt{R1R2C1C2}}{R1C1 + R2C1 + R1C2(1-K)},$$

equation (9) follows the same form as Equation (8). With some simplifications, these equations can be dealt with efficiently; the following paragraphs discuss commonly used simplification methods.

3.1   Simplification 1: Set Filter Components as Ratios

Letting R1=mR, R2=R, C1=C, and C2=nC, results in: $fc = \dfrac{1}{2\pi RC\sqrt{mn}}$ and

$Q = \dfrac{\sqrt{mn}}{m + 1 + mn(1 - K)}$. This simplifies things somewhat, but there is interaction between fc and Q. Design should start by setting the gain and Q based on m, n, and K, and then selecting C and calculating R to set fc.

Notice that $K = 1 + \dfrac{m + 1}{mn}$ results in Q = ∞. With larger values, Q becomes negative, that is, the poles move into the right half of the s-plane and the circuit oscillates. Most filters require low Q values so this should rarely be a design issue.

3.2   Simplification 2: Set Filter Components as Ratios and Gain = 1

Letting R1=mR, R2=R, C1=C, C2=nC, and K=1 results in: $fc = \dfrac{1}{2\pi RC\sqrt{mn}}$ and

$Q = \dfrac{\sqrt{mn}}{m + 1}$. This keeps gain = 1 in the pass band, but again there is interaction between fc and Q. Design should start by choosing the ratios m and n to set Q, and then selecting C and calculating R to set fc.

3.3   Simplification 3: Set Resistors as Ratios and Capacitors Equal

Letting R1=mR, R2=R, and C1=C2=C, results in: $fc = \dfrac{1}{2\pi RC\sqrt{m}}$ and

$Q = \dfrac{\sqrt{m}}{1 + 2m - mK}$. The reason for setting the capacitors equal is the limited selection of values in comparison with resistors.

There is interaction between setting fc and Q. Design should start with choosing m and K to set the gain and Q of the circuit, and then choosing C and calculating R to set fc.

3.4   Simplification 4: Set Filter Components Equal

Letting R1=R2=R, and C1=C2=C, results in: $fc = \dfrac{1}{2\pi RC}$ and $Q = \dfrac{1}{3 - K}$. Now fc and Q are independent of one another, and design is greatly simplified although limited. The gain of the circuit now determines Q. RC sets fc—the capacitor chosen and the resistor calculated. One minor drawback is that since the gain controls the Q of the circuit, further gain or attenuation may be necessary to achieve the desired signal gain in the pass band.

Values of K very close to 3 result in high Qs that are sensitive to variations in the values of R3 and R4. For instance, setting K=2.9 results in a nominal Q of 10. Worst case analysis with 1% resistors results in Q=16. Whereas, setting K=2 for a Q of 1, worst case analysis with 1% resistors results in Q=1.02. Resistor values where K=3 leads to Q=∞, and with larger values, Q becomes negative, the poles move into the right half of the s-plane, and the circuit oscillates. The most frequently designed filters require low Q values and this should rarely be a design issue.

3.5 Nonideal Circuit Operation

The previous discussions and calculations assumed an ideal circuit, but there is a frequency where this is no longer a valid assumption. Logic says that the amplifier must be an active component at the frequencies of interest or else problems occur. But what problems?

As mentioned above there are three basic modes of operation: below cut-off, above cutoff, and in the area of cutoff. Assuming the amplifier has adequate frequency response beyond cut-off, the filter works as expected. At frequencies well above cut-off, the high frequency (HS) model shown in Figure 6 is used to show the expected circuit operation. The assumption made here is that C1 and C2 are effective shorts when compared to the impedance of R1 and R2 so that the amplifier's input is at ac ground. In response, the amplifier generates an ac ground at its output limited only by its output impedance, Zo. The formula shows the transfer function of this model.
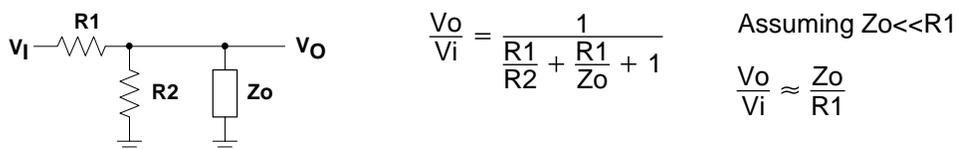
$$\frac{Vo}{Vi} = \frac{1}{\frac{R1}{R2} + \frac{R1}{Zo} + 1} \qquad \text{Assuming Zo<<R1}$$

$$\frac{Vo}{Vi} \approx \frac{Zo}{R1}$$

Figure 6. Nonideal Effect of Amplifier Output Impedance and Transfer Function

Zo is the closed-loop output impedance. It depends on the loop transmission and the open-loop output impedance, zo: $Zo = \frac{zo}{1 + a(f)b}$, where a(f)b is the loop transmission. The feedback factor, b, is constant—set by resistors R3 and R4—but the open loop gain, a(f), is dependant on frequency. With dominant pole compensation, the open-loop gain of the amplifier decreases by 20 dB/dec over the usable frequencies of operation. Assuming zo is mainly resistive (usually a valid assumption up to 100 MHz), Zo increases at a rate of 20 dB/dec. The transfer function appears to be a first order high-pass. At frequencies above 100 MHz (or so) the parasitic inductance in the output starts playing a role and the transfer function transitions to a second order high-pass. Because of stray capacitance in the circuit, at higher frequency the high-pass transfer function will also roll off.

3.6 Simulation and Lab Data

A Sallen-Key low-pass filter using the Texas Instruments THS3001 shows the effects described above. The THS3001 is a high-speed current-feedback amplifier with an advertised bandwidth of 420 MHz. No particular type of filter (i.e., Butterworth, Chebychev, Eliptic, etc.) was designed. Choosing Z1=Z2=1kΩ, Z3=Z4=1nF, R3=open, and R4=1kΩ results in a low-pass filter with fc=159 kHz, and Q=1/2.

Simulation using the spice model of the THS3001 (see the application note *Building a Simple SPICE Model for the THS3001*, SLOA018) is used to show the expected behavior of the circuit. Figure 7 shows the simulation circuits and the lab circuit tested. The results are plotted in Figure 8.

Figure 7 a) shows the simulation circuit with the spice model modified so that the output impedance is zero. Curve a) in Figure 8 shows the frequency response as simulated in spice. It shows that without the output impedance the attenuation of the signal continues to increase as frequency increases.

Figure 7 b) shows the high-frequency model shown in Figure 6 where the input is at ground and the output impedance controls the transfer function. The spice model used for the THS3001 includes the complex LRC network for the output impedance as described in the application note. Curve b) in Figure 8 shows the frequency response as simulated in spice. The magnitude of the signal at the output is seen to cross curve a) at about 7 MHz. Above this frequency the output impedance causes the switch in transfer function as described above.

Figure 7 c) shows the simulation circuit using the full spice model with the complex LCR output impedance. Curve c) in Figure 8 shows the frequency response. It shows that with the output impedance the attenuation caused by the circuit follows curve a) until it crosses curve b) at which point it follows curve b).

Figure 7 d) shows the circuit as tested in the lab, with curve d) in Figure 8 showing that the measured data agrees with the simulated data.
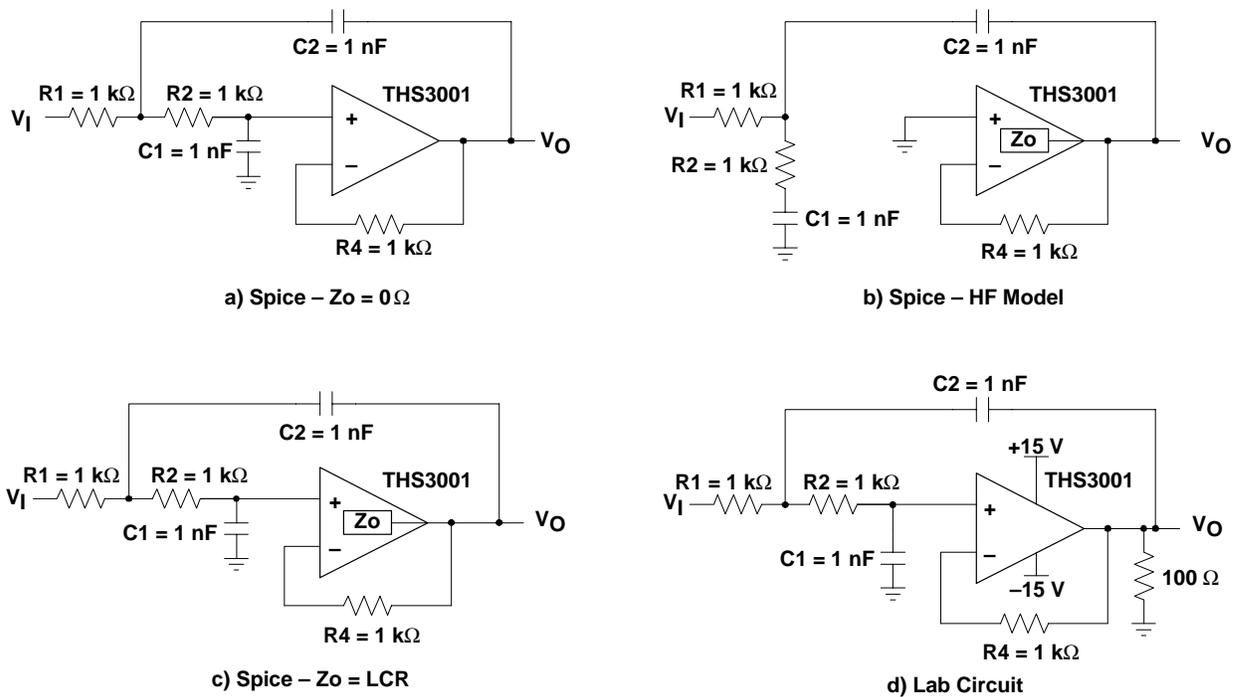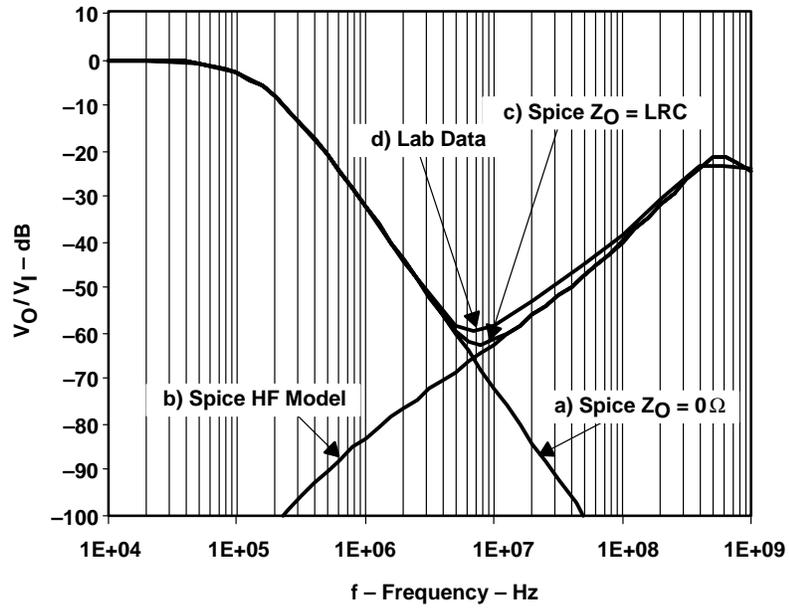
Figure 7. Test Circuits

Figure 8. Effect of Output Impedance

4   High-Pass Circuit

The standard equation (in frequency domain) for a second order high-pass is:

$$H_{HP} = \frac{-K\left(\frac{f}{fc}\right)^2}{-\left(\frac{f}{fc}\right)^2 + \frac{jf}{Qfc} + 1} \tag{10}$$

When f<<fc, equation (10) reduces to $-K\left(\frac{f}{fc}\right)^2$. Below fc signals are attenuated by the square of the frequency ratio. When f=fc, equation (10) reduces to –jKQ, and signals are enhanced by the factor Q. When f >>fc, equation (10) reduces to K, and the circuit passes signals multiplied by the gain factor K. With attenuation at lower frequencies increasing by a power of 2, equation (10) describes a second order high-pass filter.

Figure 9 shows the Sallen-Key circuit configured for high-pass:

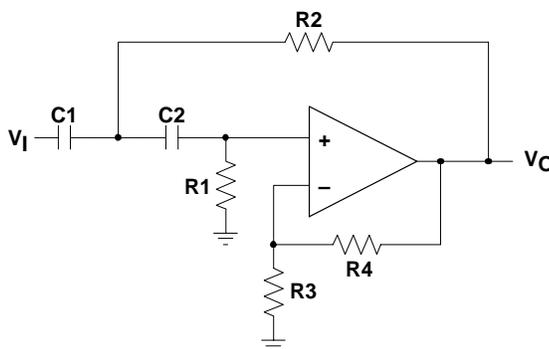$$Z2 = \frac{1}{sC2}, \ \ Z1 = \frac{1}{sC1}, \ \ Z3 = R1, \ \ Z4 = R2, \ \text{and } K = 1 + \frac{R4}{R3}.$$

Figure 9. High-Pass Sallen-Key Circuit

From equation (7), the ideal high-pass transfer function is:

$$\frac{Vo}{Vi}(hp) = \frac{K}{\frac{1}{s^2(R1R2C1C2)} + \frac{1}{s}\left(\frac{1}{R1C1} + \frac{1}{R1C2} + \frac{(1-K)}{R2C1}\right) + 1}$$

with some manipulation this becomes

$$\frac{Vo}{Vi}(hp) = \frac{K(s^2(R1R2C1C2))}{s^2(R1R2C1C2) + s(R2C2 + R2C1 + R1C2(1-K)) + 1} \tag{11}$$

By letting

$$s = j2\pi f, \ \ f_o = \frac{1}{2\pi\sqrt{R1R2C1C2}}, \ \text{and } Q = \frac{\sqrt{R1R2C1C2}}{R2C2 + R2C1 + R1C2(1-K)},$$

equation (11) follows the same form as equation (10). As above, simplifications make these equations much easier to deal with. The following are common simplifications used.

4.1 Simplification 1: Set Filter Components as Ratios

Letting R1=mR, R2=R, C1=C, and C2=nC, results in:

$$fc = \frac{1}{2\pi RC\sqrt{mn}} \text{ and } Q = \frac{\sqrt{mn}}{n + 1 + mn(1 - K)}.$$ This simplifies things somewhat, but there is interaction between fc and Q. To design a filter using this simplification, first set the gain and Q based on m, n, and K, and then select C and calculate R to set fc.

Notice that $K = 1 + \frac{n + 1}{mn}$ results in Q=∞. With larger values, Q becomes negative—that is the poles move into the right half of the s-plane and the circuit oscillates. The most frequently designed filters require low Q values and this should rarely be a design issue.

4.2 Simplification 2: Set Filter Components as Ratios and Gain=1

Letting R1=mR, R2=R, C1=C, and C2=nC, and K=1 results in:

$$fc = \frac{1}{2\pi RC\sqrt{mn}} \text{ and } Q = \frac{\sqrt{mn}}{n + 1}.$$ This keeps the gain=1 in the pass band, but again there is interaction between fc and Q. To design a filter using this simplification, first set Q by selecting the ratios m and n, and then select C and calculate R to set fc.

4.3 Simplification 3: Set Resistors as Ratios and Capacitors Equal

Letting R1=mR, R2=R, and C1=C2=C, results in:

$$fc = \frac{1}{2\pi RC\sqrt{m}} \text{ and } Q = \frac{\sqrt{m}}{2 + m(1 - K)}.$$ The reason for setting the capacitors equal is the limited selection of values compared with resistors.

There is interaction between setting fc and Q. Start the design by choosing m and K to set the gain and Q of the circuit, and then choose C and calculating R to set fc.

4.4 Simplification 4: Set Filter Components as Equal

Letting R1=R2=R, and C1=C2=C, results in: $fc = \frac{1}{2\pi RC}$ and $Q = \frac{1}{3 - K}$. Now fc and Q are independent of one another, and design is greatly simplified. The gain of the circuit now determines Q. The choice of RC sets fc—the capacitor should chosen and the resistor calculated. One minor drawback is that since the gain controls the Q of the circuit, further gain or attenuation may be necessary to achieve the desired signal gain in the pass band.

Values of K very close to 3 result in high Qs that are sensitive to variations in the values of R3 and R4. For instance, setting K=2.9 results in a nominal Q of 10. Worst case analysis with 1% resistors results in Q=16. Whereas, setting K=2 for a Q of 1, worst case analysis with 1% resistors results in Q=1.02. Resistor values where K=3 leads to Q=∞, and with larger values, Q becomes negative. The most frequently designed filters require low Q values, and this should rarely be a design issue.

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As mentioned above there are three basic modes of operation: below cut-off, above cutoff, and in the area of cut-off. Assuming the amplifier has adequate frequency response beyond cutoff, the filter works as expected. At frequencies well above cut-off, the high frequency (HS) model shown in Figure 10 is used to show the expected circuit operation. The assumption made here is that C1 and C2 are effective shorts when compared to the impedance of R1 and R2. The formula shows the transfer function of this model, where a(f) is the open loop gain of the amplifier and b is the feedback factor. The circuit operates as expected until $\dfrac{1}{a(f)b}$ is no longer much smaller than 1. After which the gain of the circuit falls off with the open loop gain of the amplifier. Because of practical limitations, designing a high-pass Sallen-Key filter results in a band-pass filter where the upper cutoff frequency is determined by the open loop response of the amplifier.
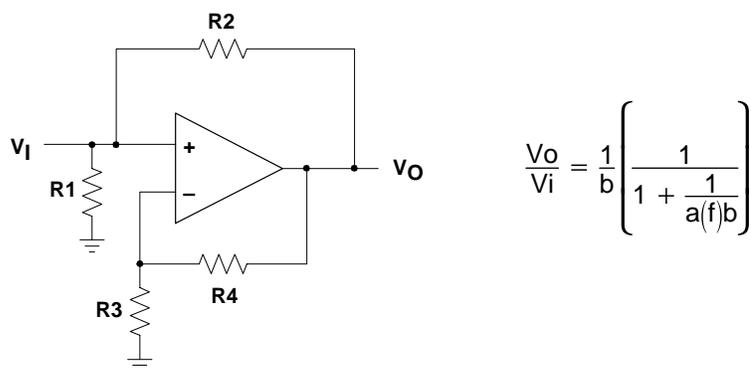

$$\frac{V_o}{V_i} = \frac{1}{b}\left[\frac{1}{1 + \dfrac{1}{a(f)b}}\right]$$

Figure 10. Model of High-Pass Sallen-Key Filter Above fc

4.6 Lab Data

A Sallen-Key high-pass filter using the Texas Instruments THS3001 shows the effects described above. The THS3001 is a high-speed current-feedback amplifier with an advertised bandwidth of 420 MHz. No particular type of filter (i.e., Butterworth, Chebychev, Eliptic, etc.) was designed. Choosing Z1=Z2=1kΩ, Z3=Z4=1nF, R3=open, and R4=1kΩ, results in a high-pass filter with fc=159 kHz, and Q=1/2. Figure 11 shows the circuit, and Figure 12 shows the lab results. As expected, the circuit attenuates signals below 159 kHz at a rate of 40dB/dec, and passes signals above 159 kHz with a gain of 1 until the amplifier's open loop gain falls to around unity between 300 MHz and 400 MHz. The slight increase in gain seen just before 300 MHz is due to gain peaking in the amplifier. Setting R4 to a higher value reduces this, but also reduces the overall bandwidth of the amplifier.
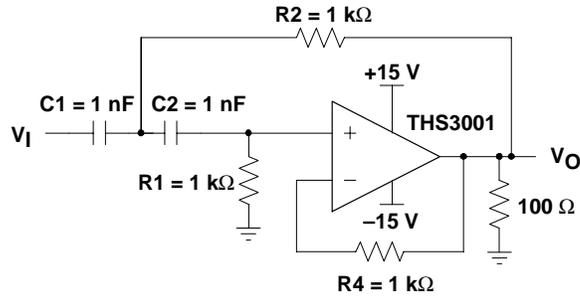
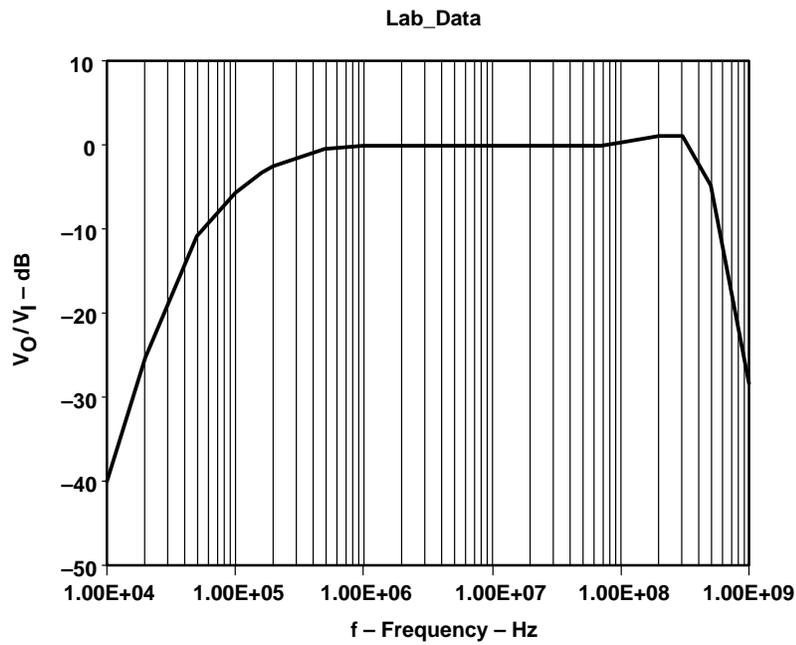**Figure 11. High-Pass Sallen-Key Filter Using THS3001**

Figure 12. Frequency Response of High-Pass Sallen-Key Filter Using THS3001

5   Summary and Comments About Component Selection

Theoretically, any values of R and C that satisfy the equations may be used, but practical considerations call for component selection guidelines to be followed.

Given a specific corner frequency, the values of C and R are inversely proportional—as C is made larger, R becomes smaller and vice versa.

In the case of the low-pass Sallen-Key filter, the ratio between the output impedance of the amplifier and the value of filter component R sets the transfer functions seen at frequencies well above cut-off. The larger the value of R the lower the transmission of signals at high frequency. Making R too large has consequences in that C may become so small that the parasitic capacitors —including the input capacitance of the amplifier—cause errors.

For the high-pass filter, the amplifier's output impedance does not play a parasitic role in the transfer function, so that the choice of smaller or larger resistor values is not so obvious. Stray capacitance in the circuit, including the input capacitance of the amplifier, makes the choice of small capacitors, and thus large resistors, undesirable. Also, being a high-pass circuit, the bandwidth is potentially very large and resistor noise associated with increased values can become an issue. Then again, small resistors become a problem if the circuit impedance is too small for the amplifier to operate properly.

The best choice of component values depends on the particulars of your circuit and the tradeoffs you are willing to make. General recommendations are as follows:

- Capacitors
  - Avoid values less than 100 pF.
  - Use NPO if at all possible. X7R is OK in a pinch. Avoid Z5U and other low quality dielectrics. In critical applications, even higher quality dielectrics like polyester, polycarbonate, mylar, etc., may be required.
  - Use 1% tolerance components. 1%, 50V, NPO, SMD, ceramic caps in standard E12 series values are available from various sources.
  - Surface mount is preferred.
- Resistors
  - Values in the range of a few hundred ohms to a few thousand ohms are best.
  - Use metal film with low temperature coefficients.
  - Use 1% tolerance (or better).
  - Surface mount is preferred.